

A BOURGAIN-PISIER CONSTRUCTION FOR GENERAL BANACH SPACES

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ABSTRACT. We prove that every Banach space, not necessarily separable, can be isometrically embedded into a \mathcal{L}_∞ -space in a way that the corresponding quotient has the Radon-Nikodym and the Schur properties. As a consequence, we obtain \mathcal{L}_∞ spaces of arbitrary large densities with the Schur and the Radon-Nikodym properties. This extends the result by J. Bourgain and G. Pisier in [Bo-Pi] for separable spaces.

1. INTRODUCTION

The main question considered in this paper is the largeness of the class of \mathcal{L}_∞ spaces in terms of embeddability. Recall that a Banach space X is called $\mathcal{L}_{\infty,\lambda}$ when for every finite dimensional subspace F of X there is a subspace G of X λ -isomorphic to $\ell_\infty^{\dim G}$ containing F . \mathcal{L}_∞ just means $\mathcal{L}_{\infty,\lambda}$ for some λ . There are two remarkable results for the class of separable \mathcal{L}_∞ spaces. The first by J. Bourgain and G. Pisier in [Bo-Pi] states that every separable Banach space X can be isometrically embedded into a \mathcal{L}_∞ -space Y_X in such a way that the corresponding quotient space Y_X/X has the Radon-Nikodym property (RNP) and the Schur property. The second, more recent one, by D. Freeman, E. Odell and Th. Schlumprecht [Fr-Od-Schl] tells that every space with separable dual can be isomorphically embedded into a \mathcal{L}_∞ -space with separable dual (therefore an ℓ_1 -predual). Both constructions are the natural extensions of the work of J. Bourgain and F. Delbaen [Bo-De] and Bourgain [Bo]. There several other recent examples. Perhaps the most impressive one is the \mathcal{L}_∞ -space by S. A. Argyros and R. G. Haydon [Ar-Ha] where every operator is the sum of a multiple of the identity and a compact one.

In the non-separable context much less is known. Spaces of functions on a non-metrizable compactum, or non-separable Gurarij spaces are non-separable \mathcal{L}_∞ -spaces. There is a wide variety of structures in the non-separable level for spaces in these two classes. Based on combinatorial axioms outside ZFC, there are non-separable spaces in these two classes without uncountable biorthogonal systems or where every operator is the sum of a multiple of the identity and an operator with separable range (see [Lo-To] for more information). On the other hand, the separable structure of the known examples is too simple: either they are c_0 -saturated, that is, every infinite dimensional subspace of it contains an isomorphic copy of c_0 , or universal for the separable spaces. So it is natural to ask if there are examples of non-separable \mathcal{L}_∞ -space without isomorphic copies of c_0 , with the (RNP) or with the Schur property. Our main result in Theorem 3.1 is that the embedding Theorem by Bourgain and Pisier remains valid for any density. In

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particular, by embedding $\ell_1(\kappa)$ for an infinite cardinal number κ , we obtain examples of \mathcal{L}_∞ spaces of arbitrary density with the Radon-Nikodym and the Schur properties.

For a given separable space X , the corresponding Bourgain-Pisier superspace Y_X of it is built in such a way that Y_X and the quotient Y_X/X are both the inductive limit of *linear* systems $(Z_n, j_n)_{n \in \mathbb{N}}$ of a special type of isometrical embedding $j_n : Z_n \rightarrow Z_{n+1}$ (η -admissible embeddings, see Definition 2.2), and such that, in addition, the corresponding Z_n 's are finite dimensional for the quotient space Y_X/X . The key fact to get the Radon-Nikodym and the Schur properties of the quotient space Y_X/X is the metric property of η -admissible embeddings exposed here in Lemma 1 and its consequence to inductive limits as above for finite dimensional spaces (see [Bo-Pi, Theorem 1.6]).

In contrast to the separable case, the main difficulty in the non-separable case is the construction the appropriate inductive limit. Indeed, if X is non-separable, then it is unlikely to find a nice linear system having the space Y_X as the corresponding limit. In general, every Banach space X is naturally represented as the inductive limit of its finite dimensional subspaces together with the corresponding inclusions between them. Our inductive system $((E_s)_{s \in I}, (j_{t,s})_{t \subseteq s})$ to represent Y_X is also based on the inclusion relation over the index set I consisting on all finite subsets of the density of the space X . This provides a natural way to isometrically embed X into Y_X . In addition, our inductive system is constructed in a way that its linear subsystems $((E_{s_n})_{n \in \mathbb{N}}, (j_{s_n, s_{n+1}})_{n \in \mathbb{N}})$ are Bourgain-Pisier linear systems as above. In other words, every separable subspace Z of Y_X can be isometrically embedded into the separable Bourgain-Pisier extension Y_Z . So, having into account that the Radon-Nikodym and Schur properties are separably determined, we readily have that the quotient Y_X/X has the desired properties.

To construct the spaces E_s and the corresponding embeddings $j_{s,t} : E_s \rightarrow E_t$ we define first finite linear systems $((E_t^{(s)})_{t \subseteq s}, (j_{u,t}^{(s)})_{u \prec t})$ of η -admissible embeddings $j_{u,t}^{(s)} : E_u^{(s)} \rightarrow E_t^{(s)}$, where \prec is a natural well ordering extending the inclusion relation. Obviously, this raises a problem of coherence, since given $s \subseteq p \subseteq q$ we will have defined two “ s -extensions” $E_s^{(p)}$ and $E_s^{(q)}$ of $E_\emptyset = X$ and therefore two isometric embeddings $X \rightarrow E_s^{(p)}$ and $X \rightarrow E_s^{(q)}$. This is corrected by defining simultaneously an infinite directed system $((E_s^{(p)})_{s \subseteq p}, (k_s^{(p,q)})_{s \subseteq p \subseteq q})$ of η -admissible embeddings making the appropriate diagrams commutative.

Finally, let us point out that nothing is known in how to skip the separability assumption in the Freeman-Odell-Schlumprecht embedding Theorem, or, even more basic, if a non-separable Bourgain-Delbaen exists, i.e. a non-separable \mathcal{L}_∞ -space not containing isomorphic copies of c_0 or ℓ_1 .

The paper is organized as follows. The Section 2 is a survey of basic facts concerning the Kisliakov's extension method and η -admissible, in particular we present a new extension fact concerning these embeddings in Lemma 2.5. The last section is devoted to the proof of the Theorem 3.1.

2. BACKGROUND AND BASIC FACTS

We use standard terminology in Banach space theory from the monographs [Al-Ka] and [Li-Tz]. The goal of this section is to present the basic notions of η -admissible diagrams and η -admissible embeddings introduced by Bourgain and Pisier. To complete the information we

give here, specially for some proofs, we refer the reader to the original paper [Bo-Pi] or to the recent book by P. Dodos [Do].

Recall the Kisliakov's extension method [Ki]: Given Banach spaces $S \subseteq B$ and E and an operator $u : S \rightarrow E$ such that $\|u\| \leq \eta \leq 1$, let

$$\begin{aligned} N_u &:= \{(s, -u(s)) \in B \times E : s \in S\}, \\ i_B &: B \rightarrow (B \oplus_1 E)/N_u \\ b &\mapsto i_B(b) = (b, 0) + N_u \\ i_E &: E \rightarrow (B \oplus_1 E)/N_u \\ e &\mapsto i_E(e) = (0, e) + N_u \end{aligned}$$

Then the diagram (K)

$$\begin{array}{ccc} B & \xrightarrow{i_B} & (B \oplus_1 E)/N_u \\ \uparrow & & \uparrow i_E \\ S & \xrightarrow{u} & E \end{array} \quad (K)$$

is commutative, i_E is an isometrical embedding, and $\|i_B\| \leq 1$. This diagram has several categorical properties such as minimality and uniqueness.

Definition 2.1. We say that a diagram

$$\begin{array}{ccc} B & \xrightarrow{\bar{u}} & E_1 \\ \uparrow & & \uparrow j \\ S & \xrightarrow{u} & E \end{array}$$

is a η -admissible diagram when there is an isometry $T : (B \oplus_1 E)/N_u \rightarrow E_1$ such that $j = T \circ i_E$ and $\bar{u} = T \circ i_B$. The *canonical* η -admissible diagram associated to the triple (S, B, u) is the Kisliakov's diagram (K) above.

An isometrical embedding $j : E \rightarrow E_1$ is called *η -admissible embedding* when there are $S \subseteq B$, E_1 , $u : S \rightarrow E$, $\bar{u} : B \rightarrow E_1$ forming together with $j : E \rightarrow E_1$ an η -admissible diagram.

Observe that η -admissible diagrams are always commutative.

Definition 2.2. [Bo-Pi] A surjective operator $\pi : E \rightarrow F$ is called a *metric surjection* when the associated isomorphism $\bar{\pi} : E/\text{Ker}(\pi) \rightarrow F$ is an isometry.

The following are useful known characterizations, not difficult to prove.

Proposition 2.3. (a) Let $S \subseteq B$, E and E_1 be normed spaces, and $\eta \leq 1$. A diagram (Δ)

$$\begin{array}{ccc} B & \xrightarrow{\bar{u}} & E_1 \\ \uparrow & (\Delta) & \uparrow j \\ S & \xrightarrow{u} & E \end{array} \quad (1)$$

is an η -admissible diagram if and only if

- ($\alpha.1$) j is an isometry and $\|u\| \leq \eta$,
- ($\alpha.2$) $\pi : B \oplus_1 E \rightarrow E_1$ defined for $(b, e) \in B \times E$ by $\pi(b, e) := \bar{u}(b) + j(e)$ is a metric surjection.
- ($\alpha.3$) $\text{Ker}(\pi) = N_u = \{(s, -u(s)) : s \in S\}$.
- (b) An isometrical embedding $j : E \rightarrow E_1$ is η -admissible iff there is some Banach space B and a metric surjection $\pi : B \oplus_1 E \rightarrow E_1$ such that $\pi(0, e) = j(e)$ for every $e \in E$ and $\|\pi(b, e)\| \geq \|e\| - \eta\|b\|$ for every $(e, b) \in E \times B$. \square

It follows from (b) above that the composition of two η -admissible embeddings is also η -admissible. Although we are not going to use them directly, two metric properties of η -admissible embeddings crucial for the Radon-Nikodym and Schur properties of the Bourgain-Pisier quotient Y_X/X .

Lemma 1. Suppose that the diagram (Δ) above is η -admissible. Then,

- (a) $\|\bar{u}(b)\| = \|(b, 0) + \text{Ker}(\pi)\| = \inf_{s \in S} \|b + s\| + \|u(s)\|$ for every $b \in B$. Consequently, $\|\bar{u}\| \leq 1$, and if there is $\delta \leq 1$ such that $\|u(s)\| \geq \delta\|s\|$ for every $s \in S$, then $\|\bar{u}(b)\| \geq \delta\|b\|$ for every $b \in B$. In other words, if u is an isomorphic embedding then so is \bar{u} with better isomorphic constant.
- (b) Let $q : E_1 \rightarrow E_1/j(E)$ be the natural quotient map. Suppose that $x_0, \dots, x_n \in E_1$ are such that $x_0 + \dots + x_n \in j(E)$. Then

$$\sum_{i=0}^n \|x_i\| \geq \left\| \sum_{i=0}^n x_i \right\| + (1 - \eta) \sum_{i=0}^n \|q(x_i)\|.$$

The fact in (b) is taken from [Do] and it has an equivalent probabilistic reformulation in [Bo-Pi]. It is the key to prove the following.

Theorem 2.4. [Bo-Pi, Theorem 1.6.] Suppose that $(E_n)_n$ is a sequence of finite dimensional spaces, and suppose that $j_n : E_n \rightarrow E_{n+1}$ is an η -admissible embedding for each n . Then the inductive limit of $(E_n, j_n)_n$ has the Schur and the Radon-Nikodym properties.

2.1. One step extension. We finish this section with the following result, somehow stating that an appropriate composition of η -admissible diagrams is again η -admissible.

Lemma 2.5. *Suppose that*

$$\begin{array}{ccccc}
 B_0 & \xrightarrow{\bar{u}_0} & X_0 & \xrightarrow{j_2} & X_2 \\
 \uparrow & & \uparrow & \swarrow u_2 & \nearrow \bar{u}_2 \\
 (\Delta_0) & & j_0 & S_1 \hookrightarrow B_1 & \\
 \downarrow & & \downarrow & \swarrow u_1 & \searrow \bar{u}_1 \\
 S_0 & \xrightarrow{u_0} & E & \xrightarrow{j_1} & X_1
 \end{array}
 \quad \begin{array}{c} \\ (\Delta.2) \\ \\ (\Delta.1) \\ \\ \end{array}
 \quad \begin{array}{c} \\ \\ \\ \\ j \end{array}
 \quad (2)$$

is a commutative diagram such that:

- (1) $(\Delta.0)$, $(\Delta.1)$ and $(\Delta.2)$ are η -admissible diagrams.
- (2) $j : X_1 \rightarrow X_2$ is an isometry.

Then the diagram

$$\begin{array}{ccc}
 B_0 & \xrightarrow{j_2 \circ \bar{u}_0} & X_2 \\
 \uparrow & & \uparrow j \\
 S_0 & \xrightarrow{j_1 \circ u_0} & X_1
 \end{array}
 \quad (3)$$

is η -admissible.

PROOF. Let $\pi : B_0 \oplus_1 X_1 \rightarrow X_2$, $\pi(b_0, x) := j_2(\bar{u}_0(b_0)) + j(x)$, and for $i = 0, 1, 2$, let $\pi_i : B_i \otimes E_i \rightarrow X_i$ be defined by $\pi_i(b, e) := \bar{u}_i(b) + j_i(e)$, where $E_0 = E_1 = E$, $E_2 = X_0$ and $B_2 = B_1$. We have to check that $(\alpha.1)$, $(\alpha.2)$ and $(\alpha.3)$ in Proposition 2.3 (a) hold. By hypothesis j is isometry and clearly $\|j_1 \circ u_0\| = \|u_0\| \leq \eta$, so we get $(\alpha.1)$.

Claim 1. $\pi(b_0, \pi_1(b_1, e)) = \pi_2(b_1, \pi_0(b_0, e))$ for every $b_0 \in B_0$, $b_1 \in B_1$ and $e \in E$.

Proof of Claim:

$$\begin{aligned}
 \pi(b_0, \pi_1(b_1, e)) &= \pi_2(b_1, \pi_0(b_0, e)) = j_2(\bar{u}_0(b_0)) + j(\pi(b_1, e)) = j_2(\bar{u}_0(b_0)) + j(\bar{u}_1(b_1) + j_1(e)) = \\
 &= j_2(\bar{u}_0(b_0) + j_0(e)) + j(\bar{u}_1(b_1)) = j_2(\bar{u}_0(b_0) + j_0(e)) + \bar{u}_2(b_1) = \\
 &= \pi_2(b_1, \bar{u}_0(b_0) + j_0(e)) = \pi_2(b_1, \pi_0(b_0, e)).
 \end{aligned}$$

□

It follows from this that π is onto.

Claim 2. $\text{Ker}(\pi) = \{(s_0, -j_1(u_0(s_0))) : s_0 \in S_0\} = \{(b_0, \pi_1(0, -u_0(b_0))) : b_0 \in S_0\}$.

Proof of Claim: The last equality follows from the fact that by definition, $\pi_1(0, -u_0(b_0)) = -j_1(u_0(b_0))$. We prove now the first equality. Fix $s_0 \in S_0$, and we work to prove that

$\pi(s_0, -j_1(u_0(s_0))) = 0$. Using the commutativity of the diagram we obtain

$$\begin{aligned}\pi(s_0, -j_1(u_0(s_0))) &= j_2(\overline{u_0}(s_0)) - j(j_1(u_0(s_0))) = j_2(\overline{u_0}(s_0)) - j_2(j_0(u_0(s_0))) = \\ &= j_2(\overline{u_0}(s_0) - j_0(u_0(s_0))) = j_2(0) = 0.\end{aligned}$$

Now suppose that $\pi(b_0, g) = 0$. Let $(b_1, e) \in B_1 \times E$ be such that $\pi_1(b_1, e) = g$. Then, by Claim 1, it follows that

$$(b_1, \pi_0(b_0, e)) \in \text{Ker}(\pi_2). \quad (4)$$

And hence, $b_1 \in S_1$ and $\pi_0(b_0, e) = -u_2(b_1)$. It follows that

$$0 = \bar{u}_0(b_0) + j_0(e) + u_2(b_1) = \bar{u}_0(b_0) + j_0(e) + j_0(u_1(b_1)) = \pi_0(b_0, e + u_1(b_1))$$

So, $b_0 \in S_0$ and $e + u_1(b_1) = -u_0(b_0)$. By applying j_1 to the last equality, we obtain that

$$g := j_1(e) + \bar{u}_1(b_1) = -j_1(u_0(b_0)),$$

as desired. \square

It follows readily that $(\alpha.3)$ holds. It rests to prove the property $(\alpha.2)$.

Claim 3. $\|\pi(b_0, g)\| = \inf_{s_0 \in S_0} \|b_0 + s_0\| + \|g - j_1(u_0(s_0))\|$.

Proof of Claim: Fix $(b_1, e) \in B_1 \times E$ such that $\pi_1(b_1, e) = g$. Then, by Claim 1 it follows that $\pi(b_0, g) = \pi_2(b_1, \pi_0(b_0, e))$. Hence,

$$\begin{aligned}\|\pi(b_0, g)\| &= \|\pi_2(b_1, \pi_0(b_0, e))\| = \|(b_1, \pi_0(b_0, e)) + \text{Ker}(\pi_2)\| = \\ &= \inf_{s_1 \in S_1} (\|b_1 + s_1\| + \|\pi_0(b_0, e) - u_2(s_1)\|) = \\ &= \inf_{s_1 \in S_1} (\|b_1 + s_1\| + \|\pi_0(b_0, e) - j_0(u_1(s_1))\|) = \\ &= \inf_{s_1 \in S_1} (\|b_1 + s_1\| + \|\pi_0(b_0, e - u_1(s_1))\|) = \\ &= \inf_{s_1 \in S_1} \left(\|b_1 + s_1\| + \inf_{s_0 \in S_0} (\|b_0 + s_0\| + \|e - u_1(s_1) - u_0(s_0)\|) \right) = \\ &= \inf_{s_0 \in S_0} \left(\|b_0 + s_0\| + \inf_{s_1 \in S_1} (\|s_1 + b_1\| + \|e - u_1(s_1) - u_0(s_0)\|) \right) = \\ &= \inf_{s_0 \in S_0} (\|b_0 + s_0\| + \|(b_1, e - u_0(s_0)) + \text{Ker}(\pi_1)\|) = \\ &= \inf_{s_0 \in S_0} (\|b_0 + s_0\| + \|\pi_1(b_1, e) + \pi_1(0, -u_0(s_0))\|) = \\ &= \inf_{s_0 \in S_0} (\|b_0 + s_0\| + \|g - j_1(u_0(s_0))\|). \end{aligned}$$

\square

From this we prove that π is a metric surjection: Fix $(b_0, g) \in B_0 \times G$, and let $(b_1, e) \in B_1 \times E$ be such that $g = \pi_1(b_1, e)$. Then by the Claim 2, it follows that

$$\begin{aligned}
\|(b_0, g) + \text{Ker}(\pi)\| &= \|(b_0, \pi_1(b_1, e)) + \text{Ker}(\pi)\| = \\
&= \inf_{s_0 \in S_0} (\|b_0 + s_0\| + \|\pi_1(b_1, e) + \pi_1(0, -u_0(s_0))\|) = \\
&= \inf_{s_0 \in S_0} (\|b_0 + s_0\| + \|\pi_1(b_1, e - u_0(s_0))\|) = \\
&= \inf_{s_0 \in S_0} \left(\|b_0 + s_0\| + \inf_{s_1 \in S_1} (\|b_1 + s_1\| + \|e - u_0(s_0) - u_1(s_1)\|) \right) = \\
&= \inf_{s_0 \in S_0} (\|b_0 + s_0\| + \|g - j_1(u_0(s_0))\|) = \|\pi(b_0, g)\|,
\end{aligned}$$

the last equality by Claim 3. \square

3. THE MAIN RESULT

Our goal is to isometrically embed a given Banach space, not necessarily separable, into a \mathcal{L}_∞ -space in such a way that the corresponding quotient has the Schur and the Radon-Nikodym properties. Extending the approach of Bourgain and Pisier, we will find the \mathcal{L}_∞ -space as a direct, not necessarily linear, limit of η -admissible embeddings. The following is our main result.

Theorem 3.1. *Every infinite dimensional Banach space X can be isometrically embedded into a \mathcal{L}_∞ -space Y of the same density that X such that the quotient Y/X has the Radon-Nikodym and the Schur properties.*

Corollary 3.2. *For every infinite cardinal number κ there is a \mathcal{L}_∞ -space of density κ with the Radon-Nikodym and the Schur properties.*

PROOF. For a fixed infinite cardinal number κ , apply the Theorem 3.1 to $X = \ell_1(\kappa)$. Then the corresponding superspace Y is the desired space, since the required properties are three space properties. \square

For the proof of Theorem 3.1 we need the following two concepts.

Definition 3.3. Recall that the *anti-lexicographical* ordering \prec on the family $[\kappa]^{<\omega}$ of finite subsets of κ is defined recursively as follows: $\emptyset \prec s$ for every non empty s , and

$$t \prec s \text{ if and only if } \begin{cases} \max t < \max s & \text{or} \\ \max t = \max s & \text{and } t \setminus \{\max t\} \prec s \setminus \{\max s\} \end{cases}$$

This is a well-ordering on $[\kappa]^{<\omega}$ that extends the inclusion relation \subsetneq . We introduce some notation: For each $\emptyset \subsetneq t \subseteq s$, we denote by $\bar{t}^{(s)}$ the immediate \prec -predecessor of t in the family $\mathcal{P}(s)$ of subsets of s , i.e.

$$\bar{t}^{(s)} := \max_{\prec} \{u \subseteq s : u \prec t\}.$$

Obviously this is well defined since $\mathcal{P}(s)$ is finite. We write \bar{t} to denote $\bar{t}^{(t)}$.

Definition 3.4. Recall that a *directed system* is $((X_i)_{i \in I}, (j_{i_0, i_1})_{i_0 \leq_I i_1})$, where X_i are Banach spaces, $<_I$ is a directed partial ordering, $j_{i_0, i_1} : X_{i_0} \rightarrow X_{i_1}$ are isometrical embeddings, such that if $i_0 \leq_I i_i \leq_I i_2$, then $j_{i_0, i_2} = j_{i_1, i_2} \circ j_{i_0, i_1}$, and such that $j_{i, i} = \text{Id}_{X_i}$.

From now on we fix an infinite dimensional Banach space X of density κ , and a dense subset $D = \{d_\alpha : \alpha < \kappa\}$ of it. For each $s \in [\kappa]^{<\omega}$, let X_s be the linear span of $\{d_\alpha\}_{\alpha \in s}$. Fix also $\lambda > 1$ and $\eta < 1$ such that $\lambda \cdot \eta < 1$.

Lemma 3.5. *There is a direct systems $((E_s)_{s \in [\kappa]^{<\omega}}, (j_{s,t})_{s \subseteq t, s, t \in [\kappa]^{<\omega}})$ and $(G_s)_{s \in [\kappa]^{<\omega}}$ such that:*

- (1) $G_s \subseteq E_s$ are Banach spaces, $E_\emptyset = X$.
- (2) Each $j_{s,t} : E_s \rightarrow E_t$ is an η -admissible isometrical embedding such that $j_{s,t}E_s$ has finite codimension in E_t .
- (3) G_s is λ -isomorphic to $\ell_\infty^{\dim G_s}$.
- (4) $\bigcup_{t \subsetneq s} j_{t,s}(G_t) \cup j_{\emptyset,s}(X_s) \subseteq G_s$.

We are ready now to give a proof of Theorem 3.1 from this lemma.

Proof of Theorem 3.1. Fix $((E_s)_{s \in [\kappa]^{<\omega}}, (j_{s,t})_{s \subseteq t, s, t \in [\kappa]^{<\omega}})$ and $(G_s)_{s \in [\kappa]^{<\omega}}$ as in Lemma 3.5. Let E be the completion of the inductive limit of $((E_s)_{s \in [\kappa]^{<\omega}}, (j_{t,s})_{t \subseteq s, t, s \in [\kappa]^{<\omega}})$. Because of property (4) in Lemma 3.5, it follows that $((G_s)_{s \in \mathcal{F}}, (j_{t,s} \upharpoonright G_t))_{t \subseteq s \in [\kappa]^{<\omega}}$ is also a directed system of finite dimensional normed spaces G_s which are λ -isomorphic to $\ell_\infty^{\dim G_s}$. Let Y be the completion of the corresponding direct limit $\lim_{s \in [\kappa]^{<\omega}} G_s$. It is clear that Y can be isometrically imbedded into E , while there is a natural isometric embedding of X into Y : X is the completion of the direct limit $((X_s)_{s \in [\kappa]^{<\omega}}, (i_{t,s})_{t \subseteq s \in [\kappa]^{<\omega}})$, where $i_{t,s} : X_t \rightarrow X_s$ is the inclusion map. For each finite subset s of κ , let $g_s : X_s \rightarrow G_s$ be $g_s := j_{\emptyset,s} \upharpoonright X_s$, which is well defined by (4). This is obviously an isometric embedding such that $j_{t,s} \circ g_s = g_s \circ i_{t,s}$ for every $t \subseteq s$, and hence X isometrically embeds into Y .

If we denote by $j_{s,\infty} : G_s \rightarrow Y$ the corresponding limit of $(j_{s,t})_{s \subseteq t}$, then $\bigcup_{s \in [\kappa]^{<\omega}} j_{s,\infty}(G_s)$ is dense in Y . It follows that Y is a $\mathcal{L}_{\infty,\lambda}$ -space. Since each G_s is finite dimensional, it follows that Y has density at most $|[\kappa]^{<\omega}| = \kappa$. Since X isometrically embeds into Y , the density of Y has to be κ . Let us see that Y/X has the Radon-Nikodym and the Schur properties: We use that Y/X is naturally isometrically embedded into E/X , and we prove that E/X has these two properties. Observe that these two properties are properties of separable subspaces of E/X . So let $Z \subseteq E/X$ be a separable subspace of E/X . By construction, we can find a sequence $(s_n)_{n \in \mathbb{N}}$ of elements of \mathcal{F} such that $s_n \subseteq s_{n+1}$, and such that Z is a subspace of the closure of the quotient

$$\left(\lim_{n \rightarrow \infty} ((E_{s_n})_{n \in \mathbb{N}}, (j_{s_n s_{n+1}})_{n \in \mathbb{N}}) \right) / X.$$

This quotient can be naturally isometrically identified with the inductive limit of finite dimensional spaces $((E_{s_n}/j_{\emptyset,s_n}X)_{n \in \mathbb{N}}, (\overline{j_{s_n, s_{n+1}}})_{n \in \mathbb{N}})$, which, by Theorem 2.4, has the two required properties. \square

The existence of the direct system in Lemma 3.5 is based on the following local construction.

Lemma 3.6. *For every finite subset s of κ there are*

$$(E_t^{(s)})_{t \subseteq s}, (G_t^{(s)})_{t \subseteq s}, (j_{u,t}^{(s)})_{u \subsetneq t, u, t \subseteq s} \text{ and } (k_u^{(t,s)})_{u \subseteq t \subseteq s} \quad (5)$$

such that

(A) (*Local directed system*) For every finite subset s of κ one has that

$$((E_t^{(s)})_{t \subseteq s}, (j_{u,t}^{(s)})_{u \prec t, u, t \subseteq s})$$

is a (finite) system of η -admissible isometrical embeddings such that $j_{u,t}^{(s)} E_u^{(s)}$ has finite codimension in $E_t^{(s)}$.

(B) (*Transition directed system*) For every finite subset u of κ one has that

$$((E_u^{(s)})_{u \subseteq s \in [\kappa]^{<\omega}}, (k_u^{(t,s)})_{u \subseteq t \subseteq s \in [\kappa]^{<\omega}})$$

is a system of η -admissible isometrical embeddings such that $k_u^{(t,s)} E_u^{(t)}$ has finite codimension in $E_u^{(s)}$.

(C) (*Coherence property*) For every $v \subseteq u \subseteq t \subseteq s$ one has that

$$k_u^{(t,s)} \circ j_{v,u}^{(t)} = j_{v,u}^{(s)} \circ k_v^{(t,s)}. \quad (6)$$

(D) For every $t \subseteq s \in [\kappa]^{<\omega}$ one has that $G_t^{(s)} \subseteq E_t^{(s)}$ is finite dimensional and

$$d(G_t^{(s)}, \ell_\infty^{\dim G_t^{(s)}}) \leq \lambda. \quad (7)$$

(E) For every $u \subseteq t \subseteq s$ one has that

$$k_u^{(t,s)}(G_u^{(t)}) = G_u^{(s)}. \quad (8)$$

(F) For every $\emptyset \subsetneq t \subseteq s$ one has that

$$j_{t,t}^{(s)}(G_t^{(s)}) \cup j_{\emptyset,t}^{(s)}(X_t) \subseteq G_t^{(s)}. \quad (9)$$

We postpone its proof, and we pass to prove Lemma 3.5.

Proof of Lemma 3.5. Fix $(E_t^{(s)})_{t \subseteq s}$, $(G_t^{(s)})_{t \subseteq s}$, $(j_{u,t}^{(s)})_{u \prec t, u, t \subseteq s}$ and $(k_u^{(t,s)})_{u \subseteq t \subseteq s}$ as in Lemma 3.6. For each finite subset s of κ set $E_s := E_s^{(s)}$ and $G_s := G_s^{(s)}$. Given $t \subseteq s$ let $j_{t,s} : E_t \rightarrow E_s$ be

$$j_{t,s} := j_{t,s}^{(s)} \circ k_t^{(t,s)}. \quad (10)$$

It follows from properties (A) and (B) in Lemma 3.6 that $j_{t,s}$ is an η -admissible embedding such that $j_{t,s} E_t$ has finite codimension in E_s .

Claim 4. $((E_s)_{s \in [\kappa]^{<\omega}}, (j_{t,s})_{t \subseteq s \in [\kappa]^{<\omega}})$ is a directed system of η -admissible isometrical embeddings.

Proof of Claim: Suppose that $u \subseteq t \subseteq s$. Then

$$\begin{aligned} j_{t,s} \circ j_{u,t} &= j_{t,s}^{(s)} \circ (k_t^{(t,s)} \circ j_{u,t}^{(t)}) \circ k_u^{(u,t)} =_{(6)} j_{t,s}^{(s)} \circ (j_{u,t}^{(s)} \circ k_u^{(t,s)}) \circ k_u^{(u,t)} = \\ &= (j_{t,s}^{(s)} \circ j_{u,t}^{(s)}) \circ (k_u^{(t,s)} \circ k_u^{(u,t)}) = j_{u,s}^{(s)} \circ k_u^{(u,s)} = j_{u,s} \end{aligned}$$

□

For each $s \in [\kappa]^{<\omega}$, let $G_s = G_s^{(s)}$.

Claim 5. (a) $d(G_s, \ell_\infty^{\dim G_s}) \leq \lambda$, i.e. G_s is λ -isomorphic to $\ell_\infty^{\dim G_s}$.

(b) For every $t \subseteq s$ one has that $j_{t,s} G_t \subseteq G_s$.

Proof of Claim: (a) follows from (7) in (D). (b): By (8) and (9) one has that

$$j_{t,s}(G_t) = j_{t,s}^{(s)} \circ k_t^{(t,s)}(G_t^{(t)}) = j_{t,s}^{(s)} G_t^{(s)} \subseteq G_s^{(s)} = G_s.$$

□

□

It only rests to give a proof of Lemma 3.6.

Proof of Lemma 3.6. Fix $s \in [\kappa]^{<\omega}$. We define \preceq -recursively on s all the objects in (5) together with an integer $n_t \in \mathbb{N}$, $S_t \subseteq \ell_\infty^{n_t}$ and $v_t : S_t \rightarrow E_{\bar{t}}^{(t)}$ for each $\emptyset \subsetneq t \subseteq s$ such that

- (a) $E_\emptyset^{(s)} = X$, $k_\emptyset^{(t,s)} = \text{Id}_X$.
- (b) $v_t : S_t \rightarrow E_{\bar{t}}^{(t)}$ is an isomorphism with $\|v_t\| \leq \eta$, $\|v_t^{-1}\| \leq \lambda$, and

$$v_t(S_t) = \langle j_{u,\bar{t}}^{(t)}(G_u^{(t)}) \cup j_{\emptyset,\bar{t}}^{(t)}(X_t) \rangle \subseteq E_{\bar{t}}^{(t)}, \quad (11)$$

where $u = \bar{\bar{t}}^{(t)}$ is the \prec -penultimate element in $\mathcal{P}(t)$, if $|t| > 1$ and $u = \emptyset$, if $|t| = 1$ (and hence $\bar{t} = \emptyset$).

- (c) $E_t^{(s)} = (\ell_\infty^{n_t} \oplus_1 E_{\bar{t}^{(s)}}^{(s)})/N_{v_t^{(s)}}$, the diagram

$$\begin{array}{ccc} \ell_\infty^{n_t} & \xrightarrow{\overline{v_t^{(s)}}} & E_t^{(s)} = (\ell_\infty^{n_t} \oplus_1 E_{\bar{t}^{(s)}}^{(s)})/N_{v_t^{(s)}} \\ \uparrow & & \uparrow j_{\bar{t}^{(s)},t}^{(s)} \\ S_t & \xrightarrow{v_t^{(s)}} & E_{\bar{t}^{(s)}}^{(s)} \\ \downarrow v_t & & \downarrow j_{\bar{t},\bar{t}^{(s)}}^{(s)} \\ E_{\bar{t}}^{(t)} & \xrightarrow{k_{\bar{t}}^{(t,s)}} & E_{\bar{t}}^{(s)} \end{array} \quad (\Delta)$$

is commutative, (Δ) is a canonical η -admissible diagram, and

$$G_t^{(s)} = \overline{v_t^{(s)}}(\ell_\infty^{n_t}) = \{(x, 0) + N_t^{(s)} : x \in \ell_\infty^{n_t}\}. \quad (12)$$

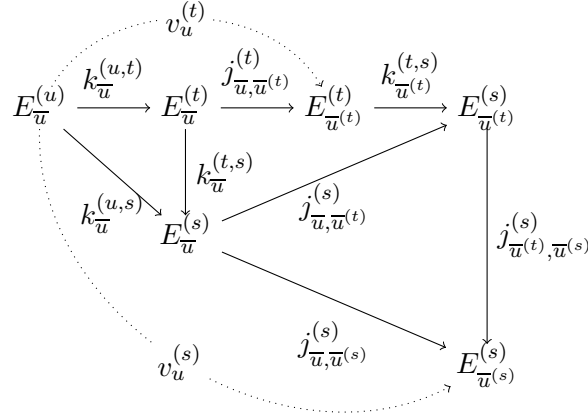
- (d) For every $u \prec t$ subset of s , we have that

$$j_{u,t}^{(s)} = j_{\bar{t}^{(s)},t}^{(s)} \circ j_{\bar{t},\bar{t}^{(s)}}^{(s)}. \quad (13)$$

- (e) For every $u \subseteq t \subseteq s$, $k_u^{(t,s)} : E_u^{(t)} \rightarrow E_u^{(s)}$ satisfies that for every $(x, y) \in \ell_\infty^{n_u} \times E_{\bar{u}(t)}^{(t)}$ by

$$k_u^{(t,s)}((x, y) + N_u^{(t)}) = (x, j_{\bar{u}(t),\bar{u}(s)}^{(s)} \circ k_{\bar{u}(t)}^{(t,s)}(y)) + N_u^{(s)}. \quad (14)$$

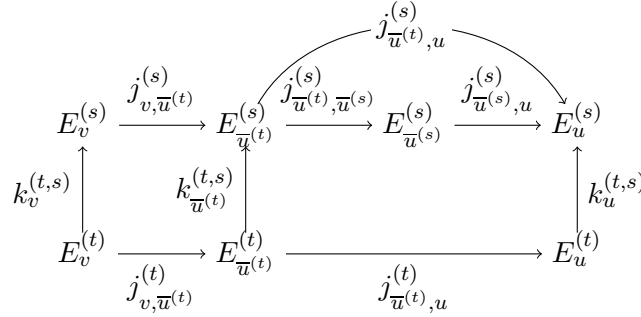
The requirement in (e) can be fulfilled because the commutativity of the following diagram:



It rests to check that the conditions (A)–(F) hold:

(A): It is clear from the definition of $j_{\bar{u}^{(t)}, t}^{(s)}$ is an η -admissible isometrical embedding, and it follows from the equality in (13) that $j_{u, t}^{(s)} = j_{v, t}^{(s)} \circ j_{u, v}^{(s)}$ for every $u, v \subseteq s$ with $u \prec v \prec t$.

(C): Let $v \subsetneq u \subseteq t \subseteq s$. We want to prove that $k_u^{(t,s)} \circ j_{v, u}^{(t)} = j_{v, u}^{(s)} \circ k_v^{(t,s)}$. Using that, by inductive hypothesis, the left side of the following diagram is commutative,



it suffices to prove that $k_u^{(t,s)} \circ j_{\bar{u}^{(t)}, u}^{(t)} = j_{\bar{u}^{(t)}, u}^{(s)} \circ k_{\bar{u}^{(t)}}^{(t,s)}$. Let $x \in E_{\bar{u}^{(t)}}^{(t)}$. Then by (e),

$$\begin{aligned} k_u^{(t,s)} \circ j_{\bar{u}^{(t)}, u}^{(t)}(x) &= k_u^{(t,s)}((0, x) + N_u^{(t)}) = (0, j_{\bar{u}^{(t)}, \bar{u}^{(s)}}^{(s)} \circ k_{\bar{u}^{(t)}}^{(t,s)}(x)) + N_u^{(s)} \\ j_{\bar{u}^{(t)}, u}^{(s)} \circ k_{\bar{u}^{(t)}}^{(t,s)}(x) &= j_{\bar{u}^{(s)}, u}^{(s)} \circ j_{\bar{u}^{(t)}, \bar{u}^{(s)}}^{(s)} \circ k_{\bar{u}^{(t)}}^{(t,s)}(x) = (0, j_{\bar{u}^{(t)}, \bar{u}^{(s)}}^{(s)} \circ k_{\bar{u}^{(t)}}^{(t,s)}(x)) + N_u^{(s)}. \end{aligned}$$

(B): Suppose that $v \subseteq u \subseteq t \subseteq s$. We have to see that $k_v^{(u,s)} = k_v^{(t,s)} \circ k_v^{(u,t)}$. Recall that from (e) it follows that for every $(x, y) \in \ell_\infty^{n_v} \times E_{\bar{v}^{(u)}}$ and for every $(x, z) \in \ell_\infty^{n_v} \times E_{\bar{v}^{(t)}}$ one has that

$$k_v^{(u,t)}((x, y) + N_v^{(u)}) = (x, j_{\bar{v}^{(u)}, \bar{v}^{(t)}}^{(t)} \circ k_{\bar{v}^{(u)}}^{(u,t)}(y)) + N_v^{(t)} \quad (15)$$

$$k_v^{(u,s)}((x, y) + N_v^{(u)}) = (x, j_{\bar{v}^{(u)}, \bar{v}^{(s)}}^{(s)} \circ k_{\bar{v}^{(u)}}^{(u,s)}(y)) + N_v^{(s)} \quad (16)$$

$$k_v^{(t,s)}((x, z) + N_v^{(t)}) = (x, j_{\bar{v}^{(t)}, \bar{v}^{(s)}}^{(s)} \circ k_{\bar{v}^{(t)}}^{(t,s)}(z)) + N_v^{(s)}. \quad (17)$$

Hence, using inductively (C),

$$\begin{aligned}
k_v^{(t,s)} \circ k_v^{(u,t)}((x, y) + N_v^{(u)}) &= (x, j_{\bar{v}(t), \bar{v}(s)}^{(s)} \circ k_{\bar{v}(t)}^{(t,s)} \circ j_{\bar{v}(u), \bar{v}(t)}^{(t)} \circ k_{\bar{v}(u)}^{(u,t)}(y)) + N_v^{(t)} = \\
&= (x, j_{\bar{v}(t), \bar{v}(s)}^{(s)} \circ j_{\bar{v}(u), \bar{v}(t)}^{(s)} \circ k_{\bar{v}(u)}^{(t,s)} \circ k_{\bar{v}(u)}^{(u,t)}(y)) + N_v^{(t)} \\
&= (x, j_{\bar{v}(u), \bar{v}(s)}^{(s)} \circ k_{\bar{v}(u)}^{(u,s)}(y)) + N_v^{(t)} \\
&= k_v^{(u,s)}((x, y) + N_v^{(u)}).
\end{aligned}$$

We now prove that $k_u^{(t,s)}$ is an η -admissible isometrical embedding: By inductive hypothesis the composition $j : E_{\bar{u}(t)}^{(t)} \rightarrow E_{\bar{u}(s)}^{(s)}$, $j := j_{\bar{u}(t), \bar{u}(s)}^{(s)} \circ k_{\bar{u}(t)}^{(t,s)}$, is an η -admissible isometrical embedding. We then fix $S \subseteq B$, $\nu : S \rightarrow E_{\bar{u}(t)}^{(t)}$ and $\bar{\nu} : B \rightarrow E_{\bar{u}(s)}^{(s)}$ such that

$$\begin{array}{ccc}
B & \xrightarrow{\bar{\nu}} & E_{\bar{u}(s)}^{(s)} \\
\uparrow & (\Delta_0) & \uparrow j = j_{\bar{u}(t), \bar{u}(s)}^{(s)} \circ k_{\bar{u}(t)}^{(t,s)} \\
S & \xrightarrow{\nu} & E_{\bar{u}(t)}^{(t)}
\end{array}$$

is an η -admissible diagram. It follows by (C), that the following diagram is commutative:

$$\begin{array}{ccccc}
B & \xrightarrow{\bar{u}_0} & E_{\bar{u}(s)}^{(s)} & \xrightarrow{j_{\bar{u}(s), u}^{(s)}} & E_u^{(s)} = (\ell_\infty^{n_u} \oplus_1 E_{\bar{u}(s)}^{(s)})/N_u^{(s)} \\
\uparrow & & \uparrow j & \swarrow v_u^{(s)} \quad (\Delta.2) & \searrow \overline{v_u^{(s)}} \\
S & \xrightarrow{u_0} & E_{\bar{u}(t)}^{(t)} & \xrightarrow{j_{\bar{u}(t), u}^{(t)}} & E_u^{(t)} = (\ell_\infty^{n_u} \oplus_1 E_{\bar{u}(t)}^{(t)})/N_u^{(t)} \\
& & & \swarrow v_u^{(t)} \quad (\Delta.1) & \searrow \overline{v_u^{(t)}} \\
& & & & \uparrow k_u^{(t,s)}
\end{array}$$

Since (Δ_0) , (Δ_1) and (Δ_2) are η -admissible diagram, we conclude from Lemma 2.5 that $k_u^{(t,s)}$ is an η -admissible embedding.

(D) is clear by definition of $G_t^{(s)}$.

(E): Fix $u \subseteq t \subseteq s$. Then

$$k_u^{(t,s)}(G_u^{(t)}) = \{k_u^{(t,s)}((x, 0) + N_u^{(t)}) : x \in \ell_\infty^{n_u}\} = \{(x, 0) + N_u^{(s)} : x \in \ell_\infty^{n_u}\} = G_u^{(s)}.$$

(F): Let $t \subseteq s$. We have to prove the inclusion in (9). Notice that the diagram

$$\begin{array}{ccc}
 \ell_\infty^{n_t} & \xrightarrow{\overline{v_t^{(s)}}} & E_t^{(s)} = (\ell_\infty^{n_t} \oplus_1 E_{\overline{u}^{(s)}}^{(s)})/N_t^{(s)} \\
 \uparrow & & \uparrow j_{\overline{t}^{(s)},t}^{(s)} \\
 S_t & \xrightarrow{v_t^{(s)}} & E_{\overline{u}^{(t)}}^{(s)}
 \end{array}$$

is commutative. Let $u \subseteq t$ be the immediate \prec -predecessor of \overline{t} in t , if $|t| > 1$, and let $u = \emptyset$ otherwise. Then by (b),

$$\begin{aligned}
 G_t^{(s)} &= \{(x, 0) + N_t^{(s)} : s \in S_t\} = \overline{v_t^{(s)}}(\ell_\infty^{n_t}) \supseteq \\
 &\supseteq \overline{v_t^{(s)}}(S_t) = j_{\overline{t}^{(s)},t}^{(s)} \circ v_t^{(s)}(S_t) = \\
 &= j_{\overline{t}^{(s)},t}^{(s)} \circ j_{\overline{t},\overline{t}^{(s)}}^{(s)} \circ k_{\overline{t}}^{(t,s)} \circ v_t(S_t) = j_{\overline{t},t}^{(s)} \circ k_{\overline{t}}^{(t,s)} \circ v_t(S_t) = k_t^{(t,s)} \circ j_{\overline{t},t}^{(t)} \circ v_t(S_t) = \\
 &= k_t^{(t,s)} \circ j_{\overline{t},t}^{(t)} \left\langle j_{u,\overline{t}}^{(t)}(G_u^{(t)}) \cup j_{\emptyset,\overline{t}}^{(t)}(X_t) \right\rangle = \\
 &= \left\langle k_t^{(t,s)} \circ j_{\overline{t},t}^{(t)} \circ j_{u,\overline{t}}^{(t)}(G_u^{(t)}) \cup k_t^{(t,s)} \circ j_{\overline{t},t}^{(t)} \circ j_{\emptyset,\overline{t}}^{(t)}(X_t) \right\rangle = \\
 &= \left\langle k_t^{(t,s)} \circ j_{u,t}^{(t)}(G_u^{(t)}) \cup k_t^{(t,s)} \circ j_{\emptyset,t}^{(t)}(X_t) \right\rangle = \\
 &= \left\langle j_{u,t}^{(s)} \circ k_u^{(t,s)}(G_u^{(t)}) \cup j_{\emptyset,t}^{(s)} \circ k_{\emptyset}^{(t,s)}(X_t) \right\rangle = \\
 &= \left\langle j_{u,t}^{(s)}(G_u^{(s)}) \cup j_{\emptyset,t}^{(s)}(X_t) \right\rangle.
 \end{aligned}$$

□

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